Appendix E
Analytical Solution For Two-Dimensional Flow To A Well

E-1. Introduction

An analytical solution for two-dimensional flow to a well can be obtained by superposition of a point sink solution along the length of the well screen.

An equation for two-dimensional radial flow can be expressed as

$$\frac{\partial^2 P^2}{\partial r^2} + \frac{1}{r}\frac{\partial P^2}{\partial r} + \frac{\partial^2 P^2}{\partial z^2} = 0 \qquad \text{(E-1)}$$

where

r = the horizontal radial coordinate (equivalent to $[x^2 + y_2]^{½}$ in cartesian coordinates)

z = the vertical radial coordinate (equivalent to the vertical cartesian coordinate)

The solution to this equation for a point sink located at $r = 0$, $z = z'$ in an infinite space, is

$$P^2 - P_{atm}^2 = \frac{Q_v P^* \mu}{2\pi\ k_a} \frac{1}{\sqrt{r^2 + (z - z')^2}} \qquad \text{(E-2)}$$

where

z' = z-coordinate of the point sink

The point sink solution can be integrated with respect to z to obtain a line sink solution in an infinite space

$$P^2 - P_{atm}^2 = \frac{Q_v P^* \mu}{2\pi\ k_a (L - l)} \ln\left\{ \frac{z - l + \sqrt{r^2 + (z - l)^2}}{z - L + \sqrt{r^2 + (z - L)^2}} \right\} \qquad \text{(E-3)}$$

where

l = z-coordinate of the top of the well screen

L = z-coordinate of the bottom of the well screen

E-2. Superposition to model the effects of atmospheric and impermeable boundaries

a. The effects of atmospheric and impermeable boundaries can be simulated using the method of images. Recognizing $P^2 - P^2_{atm}$ as a LaPlace potential, an atmospheric boundary at z = 0 can be simulated by adding the potential from an image source located r = 0, z = -l to L to that from a real sink located at r = 0, z = l to L (Figure E-1)

$$P^2 - P^2_{atm} = \frac{Q_v P^* \mu}{2\pi k_a (L - l)} \ln\left(\frac{z - l + \sqrt{r^2 + (z - l)^2}}{z - L + \sqrt{r^2 + (z - L)^2}} \bullet \frac{z + L + \sqrt{r^2 + (z + L)^2}}{z + l + \sqrt{r^2 + (z + l)^2}} \right) \qquad \text{(E-4)}$$

Likewise, the water table can be simulated with an image sink/source pair located at $r = 0$, $z = 2b - L$ to $2b - l$ and $r = 0$, $z = 2b + l$ to $z = 2b + L$

$$P^2 - P^2_{atm} = \frac{Q_v P^* \mu}{2\pi k_a (L - l)} \left\{ \begin{array}{c} \ln\left(\dfrac{z - l + \sqrt{r^2 + (z - l)^2}}{z - L + \sqrt{r^2 + (z - L)^2}} \bullet \dfrac{z + L + \sqrt{r^2 + (z + L)^2}}{z + l + \sqrt{r^2 + (z + l)^2}} \right) \\ + \ln\left(\dfrac{z - 2b + L + \sqrt{r^2 + (z - 2b + L)^2}}{z - 2b + l + \sqrt{r^2 + (z - 2b + l)^2}} \bullet \dfrac{z + 2b + l + \sqrt{r^2 + (z + 2b + l)^2}}{z + 2b + L + \sqrt{r^2 + (z + 2b + L)^2}} \right) \end{array} \right\} \qquad \text{(E-5)}$$

which requires a corresponding sink/source pair at $r = 0$, $z = -2b + L$ to $-2b + l$, and $r = 0$, $z = -2b - l$ to $-2b - L$

$$P^2 - P^2_{atm} = \frac{Q_v P^* \mu}{2\pi k_a (L - l)} \left\{ \begin{array}{c} \ln\left(\dfrac{z - l + \sqrt{r^2 + (z - l)^2}}{z - L + \sqrt{r^2 + (z - L)^2}} \bullet \dfrac{z + L + \sqrt{r^2 + (z + L)^2}}{z + l + \sqrt{r^2 + (z + l)^2}} \right) \\ + \ln\left(\dfrac{z - 2b + L + \sqrt{r^2 + (z - 2b + L)^2}}{z - 2b + l + \sqrt{r^2 + (z - 2b + l)^2}} \bullet \dfrac{z - 2b - L + \sqrt{r^2 + (z - 2b - L)^2}}{z - 2b - l + \sqrt{r^2 + (z - 2b - l)^2}} \right) \\ - \ln\left(\dfrac{z + 2b - l + \sqrt{r^2 + (z + 2b - l)^2}}{z + 2b - L + \sqrt{r^2 + (z + 2b - L)^2}} \bullet \dfrac{z + 2b + l + \sqrt{r^2 + (z + 2b + l)^2}}{z + 2b + L + \sqrt{r^2 + (z + 2b + L)^2}} \right) \end{array} \right\} \qquad \text{(E-6)}$$

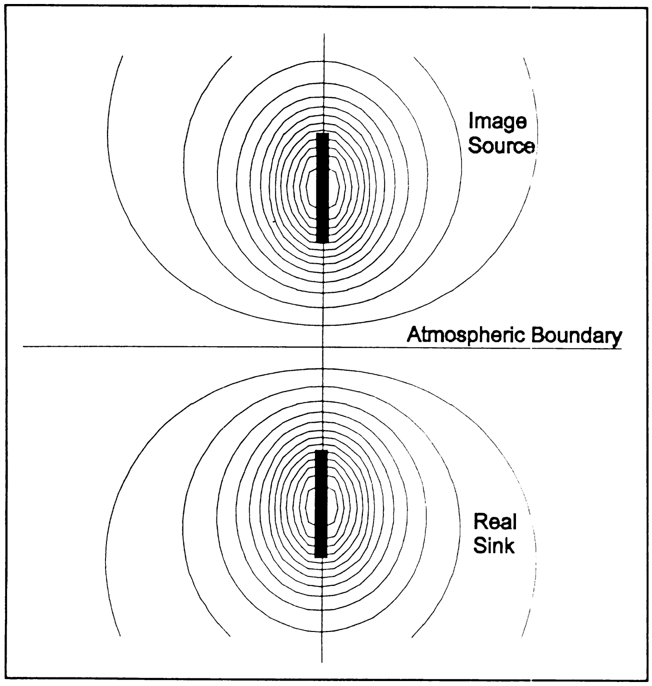

Figure E-1. Use of superposition to simulate an atmospheric boundary

More generally, each source added to balance the pressures across one boundary (e.g., the water table) produces an imbalance of pressures across the other boundary (e.g., the ground surface). As a result, additional sources and sinks are required until the incremental pressures are negligible (see Equation E-7). This is equivalent to the pressure solution obtained by Shan et al. (1992). The series summations converge in about 10 or 20 terms, and the solution can be readily evaluated on a small computer. Shan et al. (1992) provide the solution in dimensionless form, allowing application to a particular field problem through a simple scaling procedure. A plot of pressure isobars generated using Equation E-7 is shown on Figure E-2. King (1968) solved the same problem using the Dirac delta function, resulting in a slightly more complicated solution.

$$P^2 - P_{atm}^2 = \frac{Q_v P^* \mu}{2\pi k_a (L-l)} \left[\begin{array}{c} \ln\left(\dfrac{z-l+\sqrt{r^2+(z-l)^2}}{z-L+\sqrt{r^2+(z-L)^2}} \bullet \dfrac{z+L+\sqrt{r^2+(z+L)^2}}{z+l+\sqrt{r^2+(z+l)^2}} \right) \\ - \displaystyle\sum_{n=1}^{\infty} (-1)^n \ln \left(\begin{array}{c} \dfrac{z-2nb+L+\sqrt{r^2+(z-2nb+L)^2}}{z-2nb+l+\sqrt{r^2+(z-2nb+l)^2}} \bullet \dfrac{z-2nb-L+\sqrt{r^2+(z-2nb-L)^2}}{z-2nb-l+\sqrt{r^2+(z-2nb-l)^2}} \bullet \\ \dfrac{z+2nb-L+\sqrt{r^2+(z+2nb-L)^2}}{z+2nb-l+\sqrt{r^2+(z+2nb-l)^2}} \bullet \dfrac{z+2nb+L+\sqrt{r^2+(z+2nb+L)^2}}{z+2nb+l+\sqrt{r^2+(z+2nb+l)^2}} \end{array} \right) \end{array} \right] \qquad \text{(E-7)}$$

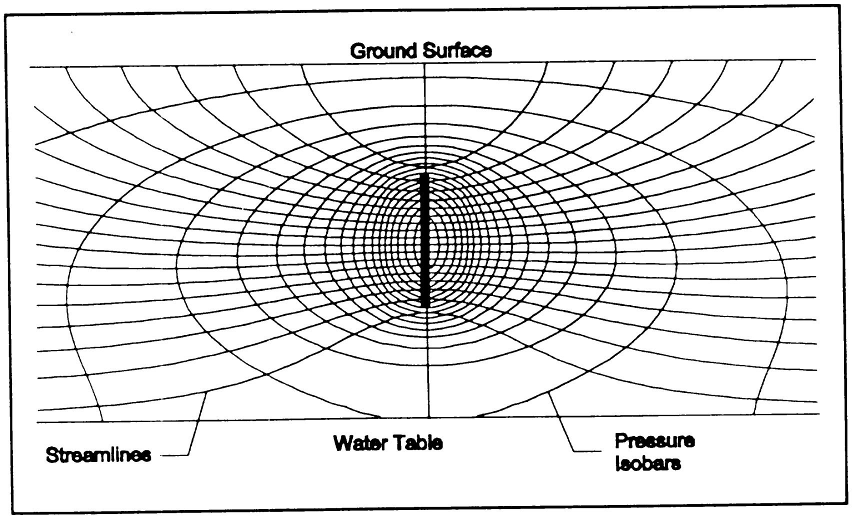

Figure E-2. Streamlines and pressure isobars

b. As described in Chapter 2, flow in anisotropic systems is governed by Equation 2-14. In order to solve this equation using the LaPlace equation (Equation 2-15), it is necessary to transform the anisotropic system into an equivalent isotropic system. This can be accomplished by choosing a coordinate system parallel to the directions of maximum and minimum air permeability (the principal directions of the air permeability tensor), and performing the coordinate transformation

$$r' = r\sqrt{\frac{k_z}{k_r}}\,; \qquad z' = z \tag{E-8}$$

Air flow equations (e.g., Equation E-7) can be solved in the transformed coordinate system using a transformed air permeability

$$k' = \sqrt{k_r \bullet k_z} \tag{E-9}$$

at which point the resulting pressure (or stream function) values can be translated back into the original coordinate system using Equations E-8.

c. The principle of superposition also permits evaluation of multiple well systems. For horizontal flow between upper and lower impermeable boundaries, the pressure distribution resulting from multiple fully penetrating wells is obtained by superposition of Equation E-10

$$P^2 - P_{atm}^2 = \sum_{i=1}^{n} \frac{Q_i\, P_i^*\, \mu}{\pi\, b\, k_a} \ln \frac{r_{e_i}}{\sqrt{(x - x_i)^2 + (y - y_i)^2}} \tag{E-10}$$

where

n = number of wells

Q_i = volumetric flow rate from the $\boldsymbol{i}^{th}$ well [L^3/T]

P_i = reference pressure for the $\boldsymbol{i}^{th}$ flow rate [M/LT^2]

r_{ei} = radius of pressure influence for the $\boldsymbol{i}^{th}$ well [L]

x_i = $\boldsymbol{x}$-coordinate of the $\boldsymbol{i}^{th}$ well

y_i = $\boldsymbol{y}$-coordinate of the $\boldsymbol{i}^{th}$ well

Similarly, for three-dimensional flow between an upper atmospheric boundary and a lower impermeable boundary, the pressure distribution resulting from multiple partially penetrating wells is obtained by superposition of Equation E-7

$$P^2 - P_{atm}^2 = \sum_{l=1}^{m} \frac{Q_i P_i^{*} \mu}{2\pi k_a (L_i - l_i)} \bullet \left\{ \ln\left[\frac{(z - l_i + \sqrt{(x-x_i)^2+(y-y_i)^2+(z-l_i)^2}}{z - L_i + \sqrt{(x-x_i)^2+(y-y_i)^2+(z-L_i)^2}} \bullet \frac{z + L_i + \sqrt{(x-x_i)^2+(y-y_i)^2+(z+L_i)^2}}{z + l_i + \sqrt{(x-x_i)^2+(y-y_i)^2+(z+l_i)^2}} \right] - \sum_{n=1}^{\infty} (-1)^n \ln\left[\frac{z - 2nb + L_i + \sqrt{(x-x_i)^2+(y-y_i)^2+(z-2nb+L_i)^2}}{z - 2nb + l_i + \sqrt{(x-x_i)^2+(y-y_i)^2+(z-2nb+l_i)^2}} \bullet \frac{z - 2nb - L_i + \sqrt{(x-x_i)^2+(y-y_i)^2+(z-2nb-L_i)^2}}{z - 2nb - l_i + \sqrt{(x-x_i)^2+(y-y_i)^2+(z-2nb-l_i)^2}} \bullet \frac{z + 2nb - L_i + \sqrt{(x-x_i)^2+(y-y_i)^2+(z+2nb-L_i)^2}}{z + 2nb - l_i + \sqrt{(x-x_i)^2+(y-y_i)^2+(z+2nb-l_i)^2}} \bullet \frac{z + 2nb + L_i + \sqrt{(x-x_i)^2+(y-y_i)^2+(z+2nb+L_i)^2}}{z + 2nb + l_i + \sqrt{(x-x_i)^2+(y-y_i)^2+(z+2nb+l_i)^2}} \right] \right\} \tag{E-11}$$

where

l_i = depth to the top of the well screen at the $\boldsymbol{i}^{\text{th}}$ well

L_i = depth to the bottom of the well screen at the $\boldsymbol{i}^{\text{th}}$ well

m = number of wells

A plot of pressure isobars generated using Equation E-11 is shown on Figure E-3.

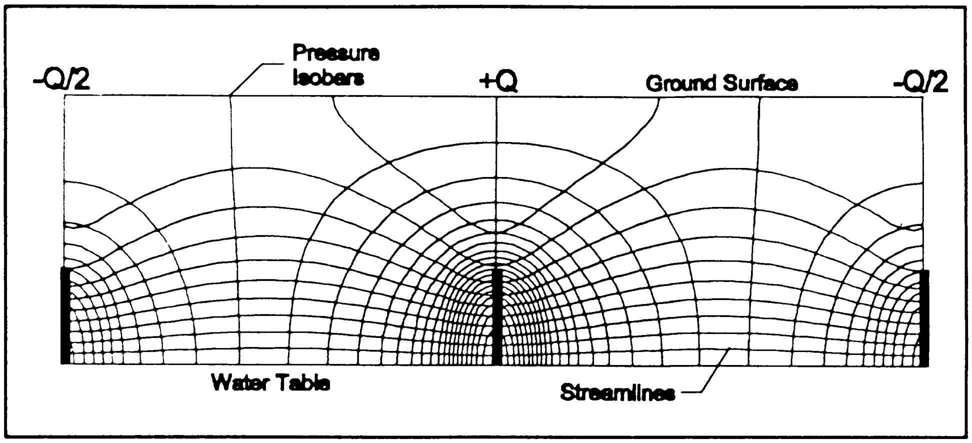

Figure E-3. Streamlines and pressure isobars for a multiwell system

d. As indicated previously (paragraph 2-4c(4)), both stream functions and potential functions satisfy the LaPlace equation. This arises from a set of equations known as the Cauchy-Rieman equations, which apply to functions satisfying the LaPlace equation. In two- dimensional Cartesian coordinates, the Cauchy-Rieman equations can be written as:

$$\frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y}; \quad \frac{\partial\phi}{\partial y} = -\frac{\partial\psi}{\partial x} \tag{E-12}$$

where

Φ = LaPlace potential

ψ = stream function

Recognizing $\boldsymbol{P}^2 - \boldsymbol{P}^2_{atm}$ as a LaPlace potential, stream functions can be obtained by performing the integration:

$$\psi = \int \frac{\partial (P^2 - P_{atm}^2)}{\partial y} dx \tag{E-13}$$

Stream functions are useful for evaluating flow paths and travel times for vapor flow. Applying Equation E-13 to the equation for one-dimensional radial flow (Equation E-10) in Cartesian coordinates yields:

$$\psi = \frac{Q_v P^* \mu}{\pi b k_a} \tan^{-1}(\frac{y - y_1}{x - x_1}) + C_1 \tag{E-14}$$

where

C_1 = a constant of integration

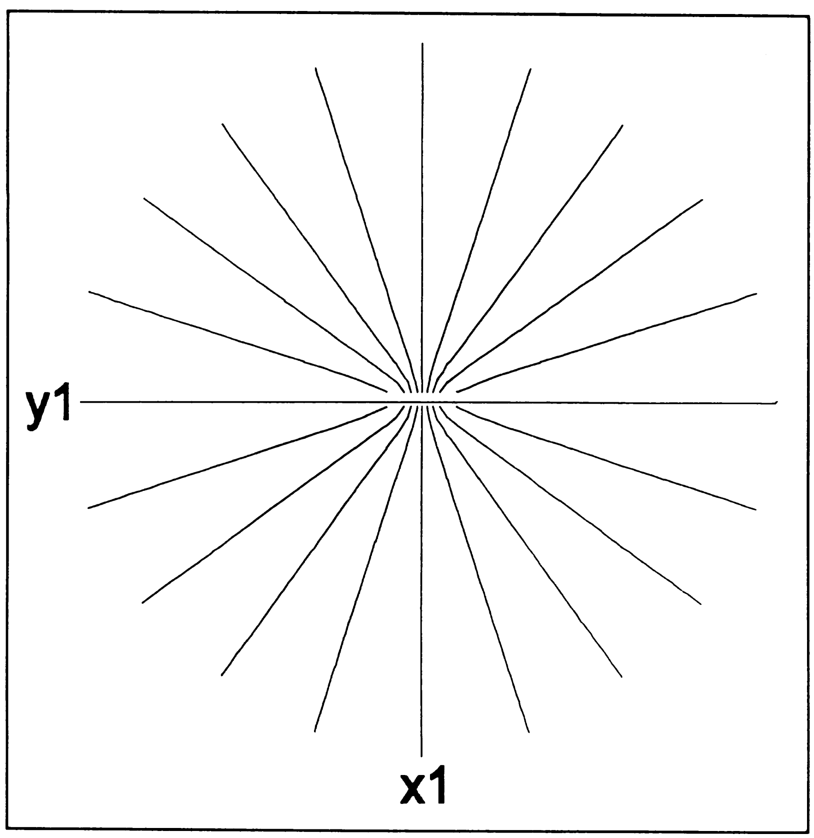

Figure E-4. Streamlines for one-dimensional radial flow

Equation E-14 represents a family of straight lines passing through (x_1,y_1), where the arctangent term is equivalent to the angle θ (in radians) between each line and the positive x-axis (Figure E-4).

Defining the angle θ as:

$$\theta = \tan^{-1} \left| \frac{y - y_1}{x - x_1} \right| \tag{E-15}$$

unique values of ψ can be specified for all **θ** by defining the constant of integration so as:

$$\begin{aligned}
\psi &= \frac{Q_v P^* \mu}{\pi\, b k_a} \theta \quad for\ 0 < \theta < \frac{\pi}{2}; \\
\psi &= \frac{Q_v P^* \mu}{\pi\, b k_a} (\pi - \theta) \quad for\ \frac{\pi}{2} < \theta < \pi; \\
\psi &= \frac{Q_v P^* \mu}{\pi\, b k_a} (\pi + \theta) \quad for\ \pi < \theta < \frac{3\pi}{2}; \\
\psi &= \frac{Q_v P^* \mu}{\pi\, b k_a} (2\pi = \theta) \quad for\ \frac{3\pi}{2} < \theta < 2\pi
\end{aligned} \tag{E-16}$$

In two-dimensional radial coordinates, the Cauchy-Rieman equations can be written as:

$$\frac{\partial\psi}{\partial z} = r \frac{\partial\phi}{\partial r}; \quad \frac{\partial\psi}{\partial r} = - r \frac{\partial\phi}{\partial z} \tag{E-17}$$

Applying Equation E-17 to the equation for two-dimensional radial flow (Equation E-7) yields:

$$\psi = \frac{Q_v P^* \mu}{2\pi\, k_a (L - l\,)}\, r \bullet$$

$$\left\{ \begin{aligned} & \left[\frac{r - (z - L) + \sqrt{r^2 + (z - L\,)^2}}{r + (z - L) + \sqrt{r^2 + (z - L\,)^2}} - \frac{r - (z - l\,) + \sqrt{r^2 + (z - l\,)^2}}{r + (z - l\,) + \sqrt{r^2 + (z - l\,)^2}} \right. \\ & \left. - \frac{r - (z + L) + \sqrt{r^2 + (z + L\,)^2}}{r + (z + L) + \sqrt{r^2 + (z + L\,)^2}} + \frac{r - (z + l\,) + \sqrt{r^2 + (z + l\,)^2}}{r + (z + l) + \sqrt{r^2 + (z + l\,)^2}} \right] \\ & - \sum_{n=1}^{\infty} (-1\,)^n \left[\frac{r - (z - 2nb + L) + \sqrt{r^2 + (z - 2nb + L\,)^2}}{r + (z - 2nb + L) + \sqrt{r^2 + (z + 2nb + L\,)^2}} - \frac{r - (z - 2nb + l\,) + \sqrt{r^2 + (z - 2nb + l\,)^2}}{r + (z - 2nb + l\,) + \sqrt{r^2 + (z - 2nb + l\,)^2}} \right. \\ & + \frac{r - (z - 2nb - L) + \sqrt{r^2 + (z - 2nb - L\,)^2}}{r + (z - 2nb - L) + \sqrt{r^2 + (z - 2nb - L\,)^2}} - \frac{r - (z - 2nb - l) + \sqrt{r^2 + (z - 2nb - l\,)^2}}{r + (z - 2nb - l\,) + \sqrt{r^2 + (z - 2nb - l\,)^2}} \\ & - \frac{r - (z + 2nb + L) + \sqrt{r^2 + (z + 2nb + L\,)^2}}{r + (z + 2nb + L) + \sqrt{r^2 + (z + 2nb + L\,)^2}} + \frac{r - (z + 2nb + l\,) + \sqrt{r^2 + (z + 2nb + l\,)^2}}{r + (z + 2nb + l\,) + \sqrt{r^2 + (z + 2nb + l\,)^2}} \\ & \left. + \frac{r - (z + 2nb - L) + \sqrt{r^2 + (z + 2nb - L\,)^2}}{r + (z + 2nb - L) + \sqrt{r^2 + (z + 2nb - L\,)^2}} - \frac{r - (z + 2nb - l\,) + \sqrt{r^2 + (z + 2nb - l\,)^2}}{r + (z + 2nb + l\,) + \sqrt{r^2 + (z + 2nb - l\,)^2}} \right] \end{aligned} \right\} \qquad \text{(E-18)}$$

Equation E-18 is equivalent to the stream function obtained by Shan, Falta, and Javandel (1990). A plot of streamlines generated using Equation E-18 is shown in Figure E-2.

As described in paragraph E-2*c*, stream functions for multiple well systems can be evaluated by superposition of Equation E-16 or E-18. A plot of streamlines for a multiple well system is shown in Figure E-3.